On Polynilpotent Covering Groups of a Polynilpotent Group

Behrooz Mashayekhy and Mahboobeh Alizadeh Sanati
Department of Mathematics, Ferdowsi University of Mashhad,
P.O.Box 1159-91775, Mashhad, Iran
E-mail: mashaf@math.um.ac.ir
sanati@math.um.ac.ir

Abstract

Let $\mathcal{N}_{c_1,\ldots,c_t}$ be the variety of polynilpotent groups of class row $(c_1,\ldots,c_t)$. In this paper, first, we show that a polynilpotent group G of class row $(c_1,\ldots,c_t)$ has no any $\mathcal{N}_{c_1,\ldots,c_t,c_{t+1}}$-covering group if its Baer-invariant with respect to the variety $\mathcal{N}_{c_1,\ldots,c_t,c_{t+1}}$ is nontrivial. As an immediate consequence, we can conclude that a solvable group G of length c with nontrivial solvable multiplier, $\mathcal{S}_nM(G)$, has no $\mathcal{S}_n$-covering group for all $n > c$, where $\mathcal{S}_n$ is the variety of solvable groups of length at most n. Second, we prove that if G is a polynilpotent group of class row $(c_1,\ldots,c_t,c_{t+1})$ such that $\mathcal{N}_{c'_1,\ldots,c'_t,c'_{t+1}}M(G) \neq 1$, where $c'_i \geq c_i$ for all $1 \leq i \leq t$ and $c'_{t+1} > c_{t+1}$, then G has no any $\mathcal{N}_{c'_1,\ldots,c'_t,c'_{t+1}}$-covering group. This is a vast generalization of the first author's result on nilpotent covering groups (Indian J. Pure Appl. Math. 29(7) 711-713, 1998).

Key words and phrases: $\mathcal{V}$-covering group; polynilpotent groups; solvable-group.

Mathematics subject classification(2000): 20E10, 20F18, 20F16

1.Introduction and Motivation

Let $G \simeq F/R$ be a free presentation for G and $\mathcal{V}$ be a variety of groups. Then, after R. Baer [1], the Baer-invariant of G with respect to $\mathcal{V}$ is defined to be $\mathcal{V}M(G) = R \cap V(F)/[RV^*F]$, where $V(F)$ is the verbal subgroup of F with respect to $\mathcal{V}$ and

$$[RV^*F] = \langle v(f_1, \cdots, f_{i-1}, f_i r, f_{i+1}, \cdots, f_n)v(f_1, \cdots, f_n)^{-1} \mid r \in R,\ f_i \in F,$$

$$1 \leq i \leq n,\ v \in V,\ n \in \mathbf{N}\rangle \ .$$

In special case, if $\mathcal{V}$ is the variety of abelian groups, then the Baer-invariant of G will be the well-known notion the Schur-multiplier of G, denoted by $M(G) = R \cap F'/[R, F]$ (See [5,6] for further details).

It is easy to see that if $\mathcal{V} = \mathcal{N}_c$, the variety of nilpotent groups of class at most $c \geq 1$, then

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,_c F]} \quad ,$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of F and $[R,_1 F] = [R, F]$, $[R,_c F] = [[R,_{c-1} F], F]$, inductively. We shall also call $\mathcal{N}_c M(G)$ the c-nilpotent multiplier of G.

In a more general case, if $\mathcal{V} = \mathcal{N}_{c_1,\ldots,c_t}$, the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$, then

$$\mathcal{N}_{c_1,\ldots,c_t} M(G) = \frac{R \cap \gamma_{c_t+1} \circ \cdots \circ \gamma_{c_1+1}(F)}{[R,_{c_1} F,_{c_2} \gamma_{c_1+1}(F), \cdots,_{c_t} \gamma_{c_{t-1}+1} \circ \cdots \circ \gamma_{c_1+1}(F)]} \quad ,$$

where $\gamma_{c_t+1} \circ \cdots \circ \gamma_{c_1+1}(F) = \gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\cdots(\gamma_{c_1+1}(F))\cdots))$ are the terms of iterated lower central series of F. See [4, corollary 6.14] for the following equality

$$[RN^*_{c_1,\ldots,c_t}F] = [R,_{c_1} F,_{c_2} \gamma_{c_1+1}(F), \ldots,_{c_t} \gamma_{c_{t-1}+1} \circ \ldots \circ \gamma_{c_1+1}(F)].$$

We shall also call $\mathcal{N}_{c_1,\ldots,c_t} M(G)$, the $(c_1, \ldots, c_t)$-polynilpotent multiplier of G.

Let $\mathcal{V}$ be a variety of groups and G be an arbitrary group, then a $\mathcal{V}$-covering group of G (a generalized covering group of G with respect to the variety $\mathcal{V}$) is a group G^* with a normal subgroup A such that $G^*/A \simeq G$, $A \subseteq V(G^*) \cap V^*(G^*)$, and $A \simeq \mathcal{V}M(G)$, where $V^*(G^*)$ is the marginal subgroup of G^* with respect to $\mathcal{V}$ (see [6]).

Note that if $\mathcal{V}$ is the variety of abelian groups, then the $\mathcal{V}$-covering group of G will be ordinary covering group (sometimes it is called representing group) of G. Also if $\mathcal{V} = \mathcal{N}_{c_1,\ldots,c_t}$, then an $\mathcal{N}_{c_1,\ldots,c_t}$-covering group of G is a group G^* with a normal subgroup A such that

$G \simeq G^*/A$,

$A \simeq \mathcal{N}_{c_1,\ldots,c_t}M(G^*)$ and

$A \subseteq N^*_{c_1,\ldots,c_t}(G^*) \cap \gamma_{c_t+1}(\cdots(\gamma_{c_1+1}(G^*))\cdots)$.

We shall also call G^* a $(c_1,\ldots,c_t)$-polynilpotent covering group of G.

It is a well-known fact that every group has at least a covering group (see [5,13]). Also, the first author proved that every group has a $\mathcal{V}$-covering group if $\mathcal{V}$ is the variety of all groups, $\mathcal{G}$, or the variety of all abelian groups, $\mathcal{A}$, or the variety of all abelian groups of exponent $m, \mathcal{A}_m$, where m is square free (see [7,9]).

Moreover, C. R. Leedham-Green and S. Mckay [6] proved, by a homological method, that a sufficient condition for existence of a $\mathcal{V}$-covering group of G is that $G/V(G)$ should be a $\mathcal{V}$-splitting group.

Some people have tried to construct a covering group for some well-known structures of groups. For example, the generalized quaternion group $Q_{4n} = \langle a,b|a^{2n}=1, b^2=a^n, b^{-1}ab=a^{-1}\rangle$ is a covering group of the dihedral group $D_{2n} = \langle a,b|a^n=b^2=1, b^{-1}ab=a^{-1}\rangle$ (see [5]).

Also J. Wiegold [12] presented a covering group for a direct product of two finite groups. W. Haebich [2, 3] generalized the Wiegold's result and gave a covering group for a regular product of a family of groups and also

for a verbal wreath product of two groups. Moreover, the first author [10] recently proved the existence and presented a structure of an $\mathcal{N}_c$-covering group for a nilpotent product of a family of cyclic groups.

It is interesting to mention that there are some groups which have no any $\mathcal{V}$-covering group, for some variety $\mathcal{V}$. The first author [8] gave an example of the group $G \simeq \mathbf{Z}_r \oplus \mathbf{Z}_s$, where $(r, s) \neq 1$, which has no $\mathcal{N}_c$-covering group for all $c \geq 2$. Moreover, the first author [7, 9] proved that a nilpotent group G of class n with nontrivial c-nilpotent multiplier $\mathcal{N}_c M(G)$, has no $\mathcal{N}_c$-covering group for all $c > n$.

Now, in this paper, we concentrate on nonexistence of polynilpotent covering groups of a polynilpotent group. More precisely, we show that if G is a polynilpotent group of class row $(c_1, \ldots, c_t)$ such that $\mathcal{N}_{c_1,\ldots,c_t,c_{t+1}} M(G) \neq 1$, then G has no $(c_1, \ldots, c_t, c_{t+1})$-polynilpotent covering group of G. Also, if $\mathcal{N}_{c'_1,\ldots,c'_t} M(G) \neq 1$ and $c'_i \geq c_i$ for all $1 \leq i \leq t-1$ and $c'_t > c_t$, then G has no $(c'_1, \ldots, c'_t)$-polynilpotent covering group of G.

2.The Main Results

Let G be a group and $\mathcal{V}$ be a variety of groups. It is clear, by definition, that if $\mathcal{V}M(G) = 1$, then G is the only $\mathcal{V}$-covering group of itself. So it is natural to put the condition $\mathcal{V}M(G) \neq 1$ for nonexistence of $\mathcal{V}$-covering group of G.

Theorem 2.1

Let G be a polynilpotent group of class row $(c_1, \ldots, c_t)$ and $\mathcal{N}_{c_1,\ldots,c_t,c_{t+1}} M(G)$ $\neq 1$, for some $c_{t+1} \geq 1$. Then G has no any $\mathcal{N}_{c_1,\ldots,c_t,c_{t+1}}$-covering group.

Proof.

Let G^* be a $(c_1, \ldots, c_t, c_{t+1})$-polynilpotent covering group of G with the normal subgroup A of G^* such that

$G \simeq G^*/A$,

$A \simeq \mathcal{N}_{c_1,\ldots,c_t,c_{t+1}} M(G^*)$ and

$A \subseteq N^*_{c_1,\ldots,c_t,c_{t+1}}(G^*) \cap \gamma_{c_{t+1}+1}(\gamma_{c_t+1}(\cdots(\gamma_{c_1+1}(G^*))\cdots))$.

We define ρ_t inductively, for any group M and $t \geq 0$, as follows:

$$\rho_0(M) = M \text{ and } \rho_i(M) = \gamma_{c_i+1}(\rho_{i-1}(M)), \text{ for } i > 1.$$

By hypothesis, $\rho_t(G) = 1$ and so $\rho_t(G^*/A) = 1$. Hence $\rho_t(G^*) \subseteq A$. Also $A \subseteq \rho_{t+1}(G^*)$, then $\rho_t(G^*) \subseteq \rho_{t+1}(G^*)$. Clearly $\rho_{t+1}(G^*) \subseteq \rho_t(G^*)$, so $\rho_{t+1}(G^*) = \rho_t(G^*)$. In particular,

$\rho_t(G^*) = \gamma_2(\rho_t(G^*)) = \cdots = \gamma_{c_{t+1}}(\rho_t(G^*)) = \gamma_{c_{t+1}+1}(\rho_t(G^*)) = \rho_{t+1}(G^*)$ (I).

Since $A \subseteq N^*_{c_1,\ldots,c_t,c_{t+1}}(G^*)$ and $\rho_t(G^*) \subseteq A$, so we have

$$[\cdots[[\rho_t(G^*),_{c_1} G^*],_{c_2} \gamma_{c_1+1}(G^*)],\cdots,_{c_{t+1}} \gamma_{c_t+1}(\cdots(\gamma_{c_1+1}(G^*))\cdots)] = 1,$$

or by the above notation,

$$[\cdots[[\rho_t(G^*),_{c_1} G^*],_{c_2} \rho_1(G^*)],\cdots,_{c_{t+1}} \rho_t(G^*)] = 1.$$

First, we show that $[M,_i N] \overset{(II)}{\supseteq} [\gamma_i(N), M]$ for each natural number i and normal subgroups M and N of any group. By Three Subgroups Lemma, we have

$[M,_i N] = [M,_{i-2} N, N, N] \supseteq [N, N, [M,_{i-2} N]] = [[M,_{i-2} N, [N, N]]$

$= [[M,_{i-3} N], N, \gamma_2(N)] \supseteq [N, \gamma_2(N), [M,_{i-3} N]] = [[M,_{i-3} N], \gamma_3(N)]$

$= [[M,_{i-4} N], N, \gamma_3(N)] \supseteq \cdots \supseteq [M, \gamma_i(N)] = [\gamma_i(N), M]$.

Now, we claim

$[\cdots[[\rho_t(G^*),_{c_1} G^*],_{c_2} \rho_1(G^*)],\cdots,_{c_i} \rho_{i-1}(G^*)] \overset{(III)}{\supseteq} [\rho_i(G^*), \rho_t(G^*)]$,

for all $1 \leq i \leq t+1$.

Clearly the equality is valid for $i = 1$. Now for $i = 2$, we can write

$[[\rho_t(G^*),_{c_1} G^*],_{c_2} \rho_1(G^*)] \supseteq [[\gamma_{c_1}(G^*), \rho_t(G^*)],_{c_2} \rho_1(G^*)] \qquad by(II)$

$\supseteq [[\rho_1(G^*), \rho_t(G^*)],_{c_2} \rho_1(G^*)]$

$\supseteq [\gamma_{c_2}(\rho_1(G^*)), [\rho_1(G^*), \rho_t(G^*)]] \qquad by(II)$

$= [[\rho_1(G^*), \rho_t(G^*)], \gamma_{c_2}(\rho_1(G^*))]$

$\supseteq [\gamma_{c_2}(\rho_1(G^*)), \rho_1(G^*), \rho_t(G^*)]$
$= [\gamma_{c_2+1}(\rho_1(G^*)), \rho_t(G^*)]$
$= [\rho_2(G^*), \rho_t(G^*)]$.
Suppose the inclusion (III) holds for $i = j$. Now, we prove it for $i = j + 1$.
$[[\cdots[[\rho_t(G^*),_{c_1} \rho_0(G^*)],_{c_2} \rho_1(G^*)], \cdots,_{c_j} \rho_{j-1}(G^*)],_{c_{j+1}} \rho_j(G^*)]$
$\supseteq [\rho_j(G^*), \rho_t(G^*),_{c_{j+1}} \rho_j(G^*)]$
$\supseteq [\gamma_{c_{j+1}}(\rho_j(G^*)), [\rho_j(G^*), \rho_t(G^*)]]$ $\quad by(I)$
$= [\rho_j(G^*), \rho_t(G^*), \gamma_{c_{j+1}}(\rho_j(G^*))]$
$\supseteq [\gamma_{c_{j+1}}(\rho_j(G^*)), \rho_j(G^*), \rho_t(G^*)]$
$= [\gamma_{c_{j+1}+1}(\rho_j(G^*)), \rho_t(G^*)]$
$= [\rho_{j+1}(G^*), \rho_t(G^*)]$.
Now, we have

$$1 = [\cdots[[\rho_t(G^*),_{c_1} \rho_0(G^*)],_{c_2} \rho_1(G^*)], \cdots,_{c_{t+1}} \rho_t(G^*)] \supseteq [\rho_{t+1}(G^*), \rho_t(G^*)].$$

Hence $[\rho_{t+1}(G^*), \rho_t(G^*)] = 1$. Since $\rho_{t+1}(G^*) = \rho_t(G^*)$, we can conclude $[\rho_t(G^*), \rho_t(G^*)] = 1$. i.e. $\gamma_2(\rho_t(G^*)) = 1$. Hence by (I), we have $\rho_{t+1}(G^*) = 1$. Therefore $A = 1$, which is a contradiction. $\square$

Now we can state the following interesting corollary about nonexistence of solvable covering groups.

Corollary 2.2

Let G be a solvable group with derived length at most n. If the l-solvable multiplier of G, $\mathcal{S}_l M(G)$, is nontrivial, then G has no any $\mathcal{S}_l$-covering group, for all $l > n$.

Proof.

Note that, $\mathcal{S}_l$, the variety of solvable groups of derived length at most l is in fact the variety of polynilpotent groups of class row $\underbrace{(1, \ldots, 1)}_{l-times}$. Hence the result is a consequence of Theorem 2.1. $\square$

In a different view, the following theorem is also about nonexistence of polynilpotent covering groups which is a vast generalization of a result of the first author (see [7, Theorem 3.1.6], [8,Theorem 2] and [9, Theorem 2.1]).

Theorem 2.3

Let G be a polynilpotent group of class row $(c_1, \ldots, c_t, c_{t+1})$ such that $\mathcal{N}_{c'_1,\ldots,c'_t,c'_{t+1}}M(G) \neq 1$ where $c'_i \geq c_i$ for all $1 \leq i \leq t$ and $c'_{t+1} > c_{t+1}$. Then G has no any $\mathcal{N}_{c'_1,\ldots,c'_t,c'_{t+1}}$-covering group.

Proof.

Let G^* be a $(c'_1, \ldots, c'_t, c'_{t+1})$-polynilpotent covering group of G with the normal subgroup A of G^* such that

$G \simeq G^*/A$,

$A \simeq \mathcal{N}_{c'_1,\ldots,c'_t,c'_{t+1}}M^*(G)$ and

$A \subseteq N^*_{c'_1,\ldots,c'_t,c'_{t+1}}(G^*) \cap \gamma_{c'_{t+1}+1}(\gamma_{c'_t+1}(\cdots(\gamma_{c'_1+1}(G^*)))\cdots))$.

We consider the following notations, inductively:

$\rho_0(G^*) = G^*$ and $\rho_i(G^*) = \gamma_{c_i+1}(\rho_{i-1}(G^*))$, for all $i \geq 1$,

$\rho'_0(G^*) = G^*$ and $\rho'_i(G^*) = \gamma_{c'_i+1}(\rho'_{i-1}(G^*))$, for all $i \geq 1$.

Since, $\rho_{t+1}(G) = 1$, so we have $\rho_{t+1}(G^*/A) = 1$, and hence $\rho_{t+1}(G^*) \subseteq A$. Also $A \subseteq \rho'_{t+1}(G^*)$, then $\rho_{t+1}(G^*) \subseteq \rho'_{t+1}(G^*)$. On the other hand, by $c'_i \geq c_i$ for all $1 \leq i \leq t$ and $c'_{t+1} > c_{t+1}$ we can imply that $\rho'_j(G^*) \subseteq \rho_j(G^*)$, for all $1 \leq j \leq t+1$. Therefore

$$\rho'_{t+1}(G^*) = \rho_{t+1}(G^*) \quad (I).$$

Consider the following trivial inclusions:

$$\rho'_{t+1}(G^*) = \gamma_{c'_{t+1}+1}(\rho'_t(G^*)) \subseteq \gamma_{c'_{t+1}}(\rho'_t(G^*)) \subseteq \gamma_{c'_{t+1}-1}(\rho'_t(G^*)) \subseteq$$

$$\cdots \subseteq \gamma_{c_{t+1}+1}(\rho'_t(G^*)) \subseteq \gamma_{c_{t+1}+1}(\rho_t(G^*)) = \rho_{t+1}(G^*).$$

Thus by the equality (I), we can conclude that

$$\gamma_{c'_{t+1}+1}(\rho'_t(G^*)) = \gamma_{c_{t+1}+1}(\rho'_t(G^*)) \ (II).$$

Since $\rho_{t+1}(G^*) \subseteq A \subseteq N^*_{c'_1,\ldots,c'_t,c'_{t+1}}(G^*)$, we have

$$[\cdots[[\rho_{t+1}(G^*),_{c'_1} \rho'_0(G^*)],_{c'_2} \rho'_1(G^*)], \cdots,_{c'_{t+1}} \rho'_t(G^*)] = 1.$$

Clearly $\rho'_t(G^*) \subseteq \rho'_i(G^*)$ for all $0 \leq i \leq t$, so by (II), we can conclude that

$$[\cdots[[\gamma_{c_{t+1}+1}(\rho'_t(G^*)),_{c'_1} \rho'_t(G^*)],_{c'_2} \rho'_t(G^*)], \cdots,_{c'_{t+1}} \rho'_t(G^*)] = 1.$$

and then $\gamma_{c_{t+1}+1+c'_1+\cdots+c'_{t+1}}(\rho'_t(G^*)) = 1$. Put $c = c_{t+1} + 1 + c'_1 + \cdots + c'_{t+1}$ and $k = c'_{t+1} - c_{t+1}$. By division algorithm, there are $q, r \in \mathbf{Z}$ such that $c = kq + r$, where $r < k$. Put $j = \min\{i \in \mathbf{N} | ki + r \geq c'_{t+1} + 1\}$. Then $kj + r \geq c'_{t+1} + 1$ and $k(j-1) + r < c'_{t+1} + 1$. Now, using (II) we have
$1 = \gamma_c(\rho'_t(G^*)) = [\gamma_{c'_{t+1}+1}(\rho'_t(G^*)),_{c-c'_{t+1}-1} \rho'_t(G^*)]$
$= [\gamma_{c_{t+1}+1}(\rho'_t(G^*)),_{c-c'_{t+1}-1} \rho'_t(G^*)]$
$= \gamma_{c-k}(\rho'_t(G^*))$
$\vdots$
$= \gamma_{c-k(q-j)}(\rho'_t(G^*))$
$= \gamma_{kj+r}(\rho'_t(G^*))$
$= \gamma_{k(j-1)+r}(\rho'_t(G^*))$
$\supseteq \gamma_{c'_{t+1}+1}(\rho'_t(G^*))$
$= \rho'_{t+1}(G^*)$. Hence $\rho'_{t+1}(G^*) = 1$ and so $A = 1$, which is a contradiction. $\square$

Notes

(i) The condition $c'_{t+1} > c_{t+1}$ in the theorem 2.3 is essential, since the first author [10] showed that for any natural number n, there exists a nilpotent group G of class n such that $\mathcal{N}_c M(G) \neq 1$ and G has at least one $\mathcal{N}_c$-covering group for all $c \leq n$.

(ii) In a joint paper with the first author [11], it is shown that a finitely generated abelian group $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \oplus \mathbf{Z}_{n_k}$, where $n_{i+1} | n_i$ for all $1 \leq i \leq k-1$, has a nontrivial polynilpotent multiplier, $\mathcal{N}_{c_1,\cdots,c_t} M(G)$, if $k \geq 3$. Hence we can find many groups satisfying in conditions of Theorems 2.1 and 2.3.

References

[1] R. Baer, Representations of Groups as Quotient Groups, I,II,III, *Trans. Amer. Math. Soc.,* **58**, (1945), 295-419.

[2] W. Haebich, The Multiplicator of a Regular Product of Groups, *Bull. Austral. Math. Soc.,* **7**, (1972), 279-296.

[3] W. Haebich, The multiplicator of a splitting extention, *Journal of Algebra,* **44**, (1977), 420-433.

[4] J. A. Hulse and J. C. Lennox, Marginal Series in Groups, *Proceedings of the Royal Society of Edinburgh,* **76**A, (1976), 139-154.

[5] G. Karpilovsky, *The Schur Multiplier,* London Math. Soc. Monographs, New Series no. **2**, (1987).

[6] C. R. Leedham-Green; S. McKay, Baer-invariant, Isologism, Varietal Laws and Homology, *Acta Math.,* **137**, (1976), 99-150.

[7] B. Mashayekhy, The Baer-invariant and Generalized Covering Groups, *Ph.D. Thesis, Ferdowsi University of Mashhad,* (1996).

[8] B. Mashayekhy, A remark on generalized covering groups, *Indian J. Pure Appl. Math.,* 29, **7**, (1998), 711-713.

[9] B. Mashayekhy, On the Existance of $\mathcal{V}$-covering groups, *Proceedings of 31-st Iranian Mathematics Conference,* (2000), pp. 227-232.

[10] B. Mashayekhy and A. Khaksar, On $\mathcal{N}_c$-Covering groups of a nilpotent product of Cyclic Groups, *International J. of Math.Game Theory and Algebra,* 13, **2**, (2003), 129-132.

[11] B. Mashayekhy and M. Parvizi, Polynilpotent Multipliers of Finitely Generated Abelian Groups, Submitted.

[12] J. Wiegold, The multiplicator of a direct product, *Quart. J. Math. Oxford,* (2), **22** (1971), 103-105.

[13] J. Wiegold, The Schur multiplier: an elementary approach pp. 137-154 in: Groups-St Andrews (1981),*London Math. Soc.* Lecture Note Ser. vol. 71, Cambridge Univ. Press (1982).